CONNECTED ABELIAN GROUPS IN COMPACT LOOPS

BY

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We shall prove the following theorem concerning compact connected abelian groups:

For any element x in a compact connected abelian group there is an element y such that the closure of the cyclic group generated by y is connected and contains x.

This is a generalisation of the well known fact that any compact connected group whose topology admits a basis of at most continuum cardinality is monothetic (i.e. contains a dense cyclic subgroup) [11]. Our proof leans heavily on the theory of duality for locally compact abelian groups which seems to be an appropriate procedure in view of the fact that the character of this theorem is typically abelian; it must remain undecided whether there is an approach using the fact that a compact connected abelian group can be approximated by torus groups.

We shall use the theorem mentioned above to determine the structure of compact loops in which every pair of elements generates an abelian subgroup. These loops are called di-associative and commutative. In a compact group the connected component of the identity is the set of all divisible elements [10]; we have been unable to establish the corresponding result for compact di-associative and commutative loops; it is easy to see that divisibility implies connectedness; as long as the converse is not established, the concept of divisibility has to replace the notion of connectivity wherever it occurs in the theory of abelian groups to obtain the analogous results for compact di-associative commutative loops. Our main result is the following:

The set D of all divisible elements in a compact di-associative commutative loop G is a closed subgroup of the center of G; no element in G/D other than the identity is divisible and G/D contains no nontrivial connected subgroup.

This implies in particular that *every compact di-associative commutative divisible loop is a group*. Moreover, G/D is a direct product of closed characteristic subloops, one for each prime p, having the property to be divisible by all natural numbers relatively prime to p. Up to the structure of these constituents the structure of compact di-associative and commutative loops is now rather completely described. One would, however, like to know whether the connected component C of the unit in G can actually be larger than the set D of divisible elements.

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I. **Monothetic subgroups of abelian groups.** In this section we shall freely use the duality theory for locally compact abelian groups; for all details we refer to [7; 11] and [12]. The following notation will be used consistently: If G is a topological group and S any subset of G then $G(S)$ denotes the closure of the subgroup generated by S. If $S=\{x\}$, we set $G(x)=G(S)$. If G is a locally compact abelian group and H a closed subgroup, then $G^\frown$ and $H^\frown$ will be the character groups of G and H, respectively, whereas H^0 denotes the subgroup of $G^\frown$ annihilated by the elements of H; in other words, if the canonical bilinear form from $G\times G^\frown$ into the group T of all complex numbers of absolute value one is denoted by $(x, y)\rightarrow\langle x, y\rangle$, then H^0 is the set of all elements y of $G^\frown$ such that $\langle x, y\rangle=1$ for all $x\in H$. The additive group of reals will be called R.

LEMMA 1.1. *If F and G are locally compact abelian groups then there is a one to one correspondence between all representations (i.e. continuous algebraic homomorphisms) $f: F\rightarrow G$ and all representations $f^\frown: G^\frown\rightarrow F^\frown$; this correspondence is fully described by the identity $\langle x, f(y)\rangle=\langle f^\frown(x), y\rangle$ for all $(x, y)\in G^\frown\times F$. The kernels and images of f and $f^\frown$ are related as follows:*

$$\ker f = (\operatorname{im} f^\frown)^0 \quad \textit{and} \quad \ker f^\frown = (\operatorname{im} f)^0.$$

This implies notably that $\operatorname{im} f=f(F)$ *is dense in G if and only if $f^\frown$ is one to one.*

LEMMA 1.2. *G contains a dense one parameter subgroup (i.e. a continuous homomorphic image of the reals) if and only if $G^\frown$ can be represented into R in a one to one fashion.*

G is monothetic (i.e. contains a dense cyclic subgroup) if and only if $G^\frown$ can be represented into T in a one to one fashion.

The proof of Lemma 1.1 belongs to the standard tools of duality and may be omitted; Lemma 1.2 is a direct consequence of 1.1. Remark that compactness of G is equivalent to discreteness of $G^\frown$; in this case $G^\frown$ can be mapped algebraically isomorphic into T if it has an isomorphic image in R, because T is algebraically isomorphic to the direct sum of R and the torsion group of T.

LEMMA 1.3. *Suppose that the compact abelian group G is the (not necessarily direct) product of two closed subgroups G_1 and G_2. The character group $G_i^\frown$ of G_i, $i=1, 2$ is isomorphic to $G^\frown/G_i^0$; let p_i be a homomorphism of $G^\frown$ onto $G_i^\frown$ with G_i^0 as kernel. Then $x\rightarrow(p_1(x), p_2(x))$ is an isomorphism of $G^\frown$ into $G_1^\frown\times G_2^\frown$.*

The first statement is well known, and the last assertion follows from the fact that the kernel of $x\rightarrow(p_1(x), p_2(x))$ is $G_1^0\cap G_2^0$ which is equal to $(G_1G_2)^0=G^0=\{1\}$.

LEMMA 1.4. *Let G be a compact abelian group, H_1 and H_2 subgroups which* *are (independently) one parameter groups or cyclic groups. Suppose that* $G=G(H_1)G(H_2)$. *Then one or the other (or both) of the following statements is true*:

(i) *G contains a cyclic subgroup H such that* $G=G(H)$.

(ii) *For* $i=1, 2$, $G_i=G(H_i)$ *contains a direct factor* G_i' *such that* G_i' *is either isomorphic to a cyclic group of prime power order* p^n *or to a p-adic group, and p is the same for* $i=1$ *and* $i=2$.

Notably, neither G_1 *nor* G_2 *can be connected.*

Proof. Let F_i denote either R or $T^{\frown}$ and let f_i be the representation of F_i onto H_i, $i=1, 2$. Let $\widehat{f_i}$ be the representation of $\widehat{G_i}$ into $\widehat{F_i}$ corresponding to f_i according to Lemma 1.1; then $\widehat{f_i}$ is one to one, since $f_i(F_i)$ is dense in $G(H_i)=G_i$. By Lemma 1.3, the mapping $g: x\to(\widehat{f_1}(p_1(x)), \widehat{f_2}(p_2(x)))$ is a one to one representation of $G^{\frown}$ into $\widehat{F_1}\times\widehat{F_2}$.

Let us now assume that statement (i) above is not true; this is the case if and only if $g(G^{\frown})$ cannot be represented in a one to one way into T. This, however, happens if and only if there is at least one p-primary torsion component T_p in $g(G^{\frown})$ which cannot be mapped isomorphically into the p-primary torsion component of T which is a Prüfer group $Z(p^\infty)$. Then the rank of T_p is at least two; on the other hand it is at most two since it is contained in the p-primary component of $\widehat{F_1}\times\widehat{F_2}$ which is isomorphic to a subgroup of $Z(p^\infty)\times Z(p^\infty)$. Thus both $\widehat{f_1}(\widehat{G_1})$ and $\widehat{f_2}(\widehat{G_2})$ must contain a nontrivial p-group as a direct factor in view of the structure of $\widehat{F_1}$ and $\widehat{F_2}$. But this, finally, is equivalent to the statement (ii) of the lemma.

In the following we have to prove a sequence of purely group theoretical lemmas. We reserve the letter N to denote the set of natural numbers. If G is a torsion free abelian group and S a subset of G, then the set $[S]$ of all elements $x\in G$, for which there is an $n\in N$ such that nx lies in the subgroup generated by S is a pure subgroup and is called the pure subgroup generated by S. By abuse of notation we write $[x]$ instead of $[\{x\}]$. Note that $[x]$ is the group generated by all roots of x: We have to observe that a is a multiple of a root of x whenever we have $ma=nx$; but, in a torsion free group, common integer factors may be cancelled; hence, without losing generality, we may assume that $(m, n)=1$; let r and s be integers which satisfy $rm+sn=1$; then $m(rx+sa)=mrx+nsx=x$, i.e., $rx+sa$ is a root of x. On the other hand, $n(rx+sa)=rma+sna=a$, which proves the assertion. We observe, furthermore, that, in a torsion free group, roots are uniquely determined. For the notion of the rank of an abelian group, see [3, p. 31].

LEMMA 1.5. *Let H be a subgroup of a torsion free abelian group G with the following properties*:

(i) G/H *is a torsion group.*

(ii) *If P is a pure subgroup of G then* $P\subset H$ *implies* $P=\{0\}$ (*i.e. H contains no nontrivial pure subgroup of G*).

Then rank $G\leq$ *rank* G/H.

Proof. We divide the proof in several steps.

(a) At first we show that rank $H=\text{rank } G$. Clearly rank $H\leqq\text{rank } G$, we have, therefore, to prove that H contains a free set of the same cardinality as the maximal free sets in G. Since we need a sharper result, we proceed to prove the following statement which implies this assertion:

(b) There is a maximal free set $\{z_i: i\in I\}$ in G with the following properties:

(ba) $z_i\notin H$,

(bb) for each $i\in I$ there is a prime number p_i such that $p_iz_i\in H$.

Let $\{x_i: i\in I\}$ be a maximal linearly independent set of G. Because of (i) for each $i\in I$ there is an $n_i\in N$ such that $n_ix_i\in H$. The set $\{n_ix_i: i\in I\}$ is clearly independent, contained in H, and maximally independent in G. Since $[n_ix_i]$ is a nontrivial pure subgroup of G, it cannot be contained in H; hence n_ix_i must have a root y_i outside H. The subgroup $H\cap G(y_i)$ of the cyclic group $G(y_i)$ is cyclic with the generator m_iy_i, say, where $m_i\neq 1$ because $y_i\notin H$. Let p_i be any prime divisor of m_i and suppose $m_i=p_im_i'$: then $m_i'y_i$ is not in H. If we define $z_i=m_i'y_i$, then (ba) and (bb) are satisfied. Since some multiple of z_i is equal to n_ix_i, the set $\{z_i: i\in I\}$ is indeed a maximal free set in G.

(c) The last step in the proof of the lemma consists in showing that the cardinality of the set $\{z_i: i\in I\}$ does not exceed the rank of G/H. The rank of G/H is the sum of the ranks of its primary components $(G/H)_p$ [3, p. 31]. The socle of a p-group is the kernel of the endomorphism $x\rightarrow px$ and is therefore a vector space over the field with p elements [3, p. 33]. The rank of a p-group is equal to the dimension of its socle. Let $\{\bar{z}_i: i\in I\}$ be the image of the set $\{z_i: i\in I\}$ under the coset homomorphism $G\rightarrow G/H$. All z_i with $p_i=p$ are in the socle of $(G/H)_p$ because of (b). We show that $\{\bar{z}_i: i\in I, p_i=p\}$ is a free set in the socle of $(G/H)_p$; then, clearly, $\{\bar{z}_i: i\in I\}$ is a free set in G/H. Suppose that some finite linear combination $\sum\{n_i\bar{z}_i: i\in I, p_i=p\}$ is zero in $(G/H)_p$. Then $\sum\{n_iz_i: i\in I, p_i=p\}\in H$; hence $n_i\equiv 0 \bmod p$ for all i under consideration which proves the assertion.

LEMMA 1.6. *Let G be a torsion free abelian group and H a subgroup of G. Then there is a subgroup P of H such that G/P is torsion free and rank $G/P\leqq$ rank G/H.*

Proof. We reduce the problem first: Choose P pure in G, contained in H, and maximal relative to these properties. Then G/P is torsion free; if, furthermore, $P\subset P'\subset H$, and P'/P is pure in G/P, then P' is pure in G [7, p. 15], and coincides with P because of the maximality of P. On considering the factor group G/P we may, from now on, assume that H contains no nontrivial subgroup which is pure in G, and we have to prove that rank $G\leqq\text{rank } G/H$.

Let A be the subgroup of G for which A/H is the torsion subgroup of G/H. Then G/A, which is isomorphic to $(G/H)/(A/H)$, is torsion free so that

A is pure in G; thus any pure subgroup of A is pure in G [7]. Therefore, H contains no nontrivial pure subgroup of A. Then, from Lemma 1.5, we have rank $A \leqq$ rank A/H. We shall show that rank $G \leqq$ rank A + rank G/A; since rank G/A = rank $(G/H)/(A/H)$ is the torsion free rank of G/H [3, pp. 31, 32], we know rank G/H = rank A/H + rank G/A. Consequently, rank $G \leqq$ rank A + rank $G/A \leqq$ rank A/H + rank G/A = rank G/H. This will finish the proof.

Let now $\{x_i: i \in I_1\}$ be a maximal free set in A so that the rank of A is the cardinality of I_1. Let $\{x_i: i \in I_2\}$ be an independent set of G such that $\{x_i: i \in I_1 \cup I_2\}$ is a maximal free set in G. Then the rank of G is the sum of the cardinalities of I_1 and I_2. We have to show that the cardinality of I_2 does not exceed rank G/A. If a finite linear combination $\sum\{n_i x_i: i \in I_2\}$ is in A, then some multiple of it is a linear combination of the x_i, $i \in I_1$ which is only possible if all n_i are zero, for $\{x_i: i \in I_1 \cup I_2\}$ is a free set. Hence the set $\{x_i: i \in I_2\}$ is free modulo A which proves the remainder.

PROPOSITION 1.7. *If H is a subgroup of a torsion free group G such that G/H is isomorphic to a subgroup of the group T of all complex numbers of absolute value* 1, *then there is a subgroup P of H such that G/P is isomorphic to a subgroup of T and is torsion free; more specifically: G/P is isomorphic to a subgroup of the reals.*

Proof. Choose P according to Lemma 1.6. Then G/P is torsion free and the rank of G/P does not exceed the rank of G/H which is at most the cardinality of the continuum. But every torsion free abelian group whose rank is less than or equal to the cardinality of the continuum has an isomorphic image in the additive group of reals since the reals are a rational vector space whose dimension is the cardinality of the continuum (see also [3, pp. 65–67]).

We are now ready to formulate the main theorem of this section:

THEOREM I. *If x is an element of a compact connected abelian group G, then there is an element $y \in G$ such that $G(y)$, the closure of the cyclic group generated by y, contains x and is connected.*

Proof. Let $G^\frown$ be the character group of G and $G(x)^0$ the annihilator of $G(x)$. Then the character group of $G(x)$, which is isomorphic to $G^\frown/G(x)^0$, is isomorphic to a subgroup of T (Lemma 1.2), and $G^\frown$ itself is torsion free because G is connected. By Proposition 1.7 there is a subgroup P of $G(x)^0$ such that $G^\frown/P$ is torsion free and isomorphic to a subgroup of R. Hence P^0 is connected in G, and contains $G(x)$ and a dense cyclic group.

REMARK. The proof of Theorem I yields also the existence of a one parameter group in G whose closure contains x.

COROLLARY I.1. *If F is a finite subset of a compact connected abelian group G, then there is an element $y \in G$ such that $G(y)$ is connected and contains F.*

Proof. If F contains only one element then the corollary is true by Theorem I. Suppose that the assertion is true if F has n elements. Then there is an element z such that $G(z)$ is connected and contains F. By the theorem, for any element x there is a z' such that $G(z')$ is connected and contains x. Then the group $G(z)G(z')$ is connected and contains the $n+1$-element set $F\cup\{x\}$; by Lemma 1.4, there is an element $y\in G$ such that $G(y)=G(z)G(z')$. Thus the corollary is proved by induction.

II. **Divisibility in compact power-associative loops.** Before we can apply the results of the preceding section to the investigation of di-associative loops, we have to provide some background material about power-associative compact loops. We recall that a loop is an algebraic structure with a binary multiplication with identity element such that the equations $ax=yb=c$ are uniquely solvable; a loop is called topological if a topology is defined on its point set and the multiplication is continuous; moreover, the above solutions x and y are to depend continuously on a and b. If every element in a loop generates a subgroup, then the loop is called power-associative. In the case of compact loops one can get away with a less restrictive assumption which is described in part of the following definition:

DEFINITION 2.1. A loop G is called mon-associative, if every element lies in a subsemigroup. If g is an element of G, then we call it divisible, if for every natural number n there is an element $x\in G$ such that $x^n=g$. The loop G itself is called divisible, if all of its elements are divisible. G is called totally indivisible if the unit is the only divisible element of G.

The following proposition indicates why it is sufficient in our context to consider mon-associative loops.

PROPOSITION 2.2. *A compact mon-associative loop is power-associative.*

Proof. Every element generates a monothetic subsemigroup whose closure is compact; it can be considered as known that this closure is actually a group. (This is, e.g., implicit in [6].)

There are locally compact connected mon-associative loops which are not power-associative [4, p. 138].

We state without proof another fact about abelian groups [7]:

PROPOSITION 2.3. *A compact abelian group is connected if and only if it is divisible.*

PROPOSITION 2.4. *In a compact divisible mon-associative loop G every element lies in a connected abelian subgroup. In particular, G is connected. Conversely, if, in a compact loop, every element lies in a connected abelian subgroup, then G is divisible.*

Proof. Let x be any element of G. Define by recursion the following sequence of elements: $x_1=x$, $x_{n+1}n+1=x_n$ for $n=1, \cdots$; this definition is possible because G is divisible. It can be seen easily that there is an algebraic homomorphism of the additive group of rationals into G such that $1/n!$ is mapped onto x_n. The image of the rational group is divisible and so is its compact closure. By Proposition 2.3, this closure is a connected compact group. Conversely, if $g\in G$ lies in a connected subgroup it is divisible in the closure of this subgroup (2.3).

The preceding proposition has the following generalisation:

PROPOSITION 2.5. *In a compact mon-associative loop G, the union D of all compact and connected abelian subgroups is compact and coincides with the set of all divisible elements of G.*

Proof. Let, for all natural numbers m, the mapping $x\rightarrow x^m$ be denoted g_m. Since $n!$ divides $(n+1)!$, the set $g_{(n+1)!}(G)$ is a subset of $g_{n!}(G)$. For some fixed natural number m and all n we have

$$g_{n!}(G) \supset g_m(g_{n!}(G)) = g_{n!m}(G) \supset g_{(nm)!}(G).$$

Therefore the descending chain $\{g_m(g_{n!}(G)): n=1, \cdots\}$ and the descending chain $\{g_{n!}(G): n=1, \cdots\}$ have the same nonvoid intersection D, which must contain all divisible elements; and the same statement holds true for all natural numbers m. Since all sets $g_{n!}(G)$ are compact, D is compact. For any continuous map $f: G\rightarrow G$ we have, by compactness, $f(D) =f(\bigcap\{g_{n!}(G): n=1, \cdots\})=\bigcap\{f(g_{n!}(G)): n=1, \cdots\}$. If we set in particular $f=g_m$, we get $g_m(D)=D$ for all natural numbers m. Hence all elements of D are divisible by elements of D. A construction analogous to the one given in the proof of 2.4 shows that every element of D lies on a compact connected subgroup which is entirely contained in D. Conversely, all compact connected subgroups are contained in D, because they are divisible and are consequently in all sets $g_{n!}(G)$.

In order to have an abbreviated terminology for the following remark, we say that a loop has property CD if the connected component C of the identity is the set D of all divisible elements; property CD thus indicates that connectivity and divisibility are equivalent concepts. We observe that, by Mycielski's result [10], all compact groups have property CD; the multiplicative loop of Cayley numbers with norm one has property CD. The additive group of a complete p-adic field is a divisible locally compact totally disconnected group. This shows that property CD can be expected only in the area of compact loops. It is now, after all, a reasonable thing to raise the following question:

Does every compact power-associative loop have property CD?

III. **Compact di-associative loops.** Throughout the rest of the paper we shall restrict our attention to a category of loops which is characterized by the following definition:

DEFINITION 3.1. A loop is called di-associative if every pair of elements generates a subgroup.

An example of a compact di-associative not commutative and not associative loop is given by the Cayley numbers of norm 1.

Similarly to the discussion of power-associative loops it is, in the compact commutative case, sufficient to make a weaker hypothesis:

PROPOSITION 3.2. *Let G be a compact commutative loop in which every pair of elements lies in a subsemigroup. Then G is di-associative.*

Proof. Let x and y be two elements of G; if H is the closure of the subsemigroup generated by x and y, then H contains the closures of the semigroups generated by x and y alone. But these closures are the groups $G(x)$ and $G(y)$ (2.2); hence the product $G(x)G(y)$ which is contained in the semigroup H is actually a group.

We are now ready to formulate the main result about compact di-associative commutative loops which exhibits the important rôle of the concept of divisibility. Let us first recall the definition of the center of a loop:

DEFINITION 3.3. The center of a loop G is the set of all elements which commute with every element of G and associate with every pair of elements of G in every order.

Intuitively, the center is the most commutative and associative part of the loop. In a di-associative commutative loop an element x is in the center if, for all pairs y, z in the loop, there is a group containing x, y and z.

THEOREM II. *Let G be a compact loop in which each pair of elements is in some commutative subsemigroup. Let D be the set of all divisible elements of G. Then D is a compact subgroup of the center of G. The factor loop G/D is totally indivisible and does not contain a connected nontrivial subgroup.*

Proof. Let $x \in D$. By 2.5, x is contained in some compact connected abelian group. By Theorem I there is an element x' in D such that $G(x')$ is connected and contains x. Let y and z be arbitrary elements of G. Then, by di-associativity, $G(x')G(y)$ is an abelian compact group. Since $G(x')$ is connected, Lemma 1.4 shows that there is an element y' such that $G(y') = G(x')G(y)$. Invoking di-associativity once more we observe that $G(y')G(z)$ is a group which contains x, y, and z. This proves that x is in the center of G. From 2.5 we know that D is a compact subset of G; in a di-associative commutative loop it is clear that D is a subloop; in our case, being in the center, D is a normal subgroup of G. Hence the factor loop G/D is well defined.

Let $\bar{a} = aD$ be a divisible element of G/D. Then, for a natural number n, there is an element $x \in G$ such that $x^n = ag$ with $g \in D$. Since D is divisible, there is an $h \in D$ such that $h^n = g$. Now $(xh^{-1})^n = x^n g^{-1} = (ag)g^{-1} = a$. Hence a is divisible and belongs, therefore, to D. Consequently $\bar{a}$ is the unit of G/D.

The fact that G/D can not contain a nontrivial connected subgroup follows from 2.3.

COROLLARY II.1. *Any compact di-associative commutative divisible loop is a group.*

COROLLARY II.2. *In any locally compact di-associative commutative loop there is a unique maximal compact divisible subloop which is, in fact, a connected compact abelian group.*

Proof. Let H^* be the closure of the set H of all elements which lie in some compact divisible subloop. Thus for each element $x \in H$ there is a $x' \in H$ such that $x \in G(x')$ and $G(x')$ is connected. By 1.4, for all x, y, and $z \in H$ the subset $(G(x')G(y'))G(z)$ is a group, because there is an element u such that $G(u) = G(x')G(y')$, and because $G(u)G(z')$ is a group on account of di-associativity. Thus H and therefore H^* is an abelian group. But every locally compact abelian group in which every element lies in a compact connected subgroup is compact [11; 12].

Clearly, H^* contains all connected compact subgroups of G, and is, also, the maximal connected compact subgroup.

Note that there are locally compact connected di-associative commutative divisible loops which are not associative [4, pp. 153–154]. If, in Corollary II.1, the hypothesis of divisibility is replaced by the assumption that G should not have small subgroups then the conclusion holds [4]; the example of a nonassociative loop mentioned before has this property, too. Compare also [8].

If the answer to the question raised at the end of §II were positive for di-associative commutative loops then Theorem II would say that the component of the identity is always in the center; in particular all connected compact di-associative commutative loops would be groups. In any event, considering the category of compact di-associative commutative loops, Theorem II gives a slight hint how strongly totally indivisible loops may possibly differ from totally disconnected ones: The former can not contain any nontrivial connected subgroups.

In view of Theorem II we may consider the structure theory of compact commutative di-associative loops as fairly complete if we uncover the structure of G/D, i.e. in general the structure of compact totally indivisible loops. The rest of the present section is devoted to this purpose.

Throughout the following, let $n_p(i)$ denote the natural number $i!$ divided by the highest possible power of p contained in it, and let e_i be the endomorphism $x \to x^{n_p(i)}$ and g_i the endomorphism $x \to x^{p^i}$. (Note that these mappings are indeed endomorphisms in di-associative commutative loops.)

PROPOSITION 3.4. *Let G be a compact totally indivisible di-associative commutative loop. We define, for each prime p, the following two sets G_p and H_p:*

$$G_p = \bigcap \{e_i(G): i = 1, \cdots \},$$

$$H_p = \bigcap \{g_i(G): i = 1, \cdots \}.$$

Then both G_p and H_p are characteristic closed subloops; G_p is divisible by all *natural numbers relatively prime to p and H_p is divisible by all powers of p. Furthermore*

$$\bigcap \{g_i(G_p): i = 1, \cdots\} = \{1\},$$
$$\bigcap \{e_i(H_p): i = 1, \cdots\} = \{1\}.$$

Proof. Since the definition and the statements are symmetric, it may be sufficient to deal with G_p alone.

If $i<j$, then $n_p(i)$ divides $n_p(j)$; therefore $e_j(G) \subset e_i(G)$. Consequently, G_p is the intersection of a descending chain of compact characteristic subloops, which shows that G_p is a compact characteristic subloop. Similar to the proof of 2.5, for all continuous mappings $f: G \rightarrow G$ we have $f(G_p) = \bigcap \{f(e_i(G)): i=1, \cdots\}$; if now f is the mapping $x \rightarrow x^m$ with a natural number m relatively prime to p, then $f(e_{im}(G)) \subset e_{im}(G) \subset f(e_i(G))$ for all $i=1, \cdots$, from which we infer $f(G_p) = G_p$; i.e. G_p is divisible by m, if m is relatively prime to p. Let now x be an element of $\bigcap \{g_i(G_p): i=1, \cdots\}$. Then x is divisible by all natural numbers m relatively prime to p because all elements of $g_i(G_p)$ have mth roots; so has x in particular, and the sequence of all these mth roots of x has a convergent subnet whose limit is in $\bigcap \{g_i(G_p): i=1, \cdots\}$ and is a mth root of x. On the other hand, x is divisible by all powers of p: Let f now be the mapping $x \rightarrow x^p$; then $f(g_i(G_p)) = g_{i+1}(G_p)$ and $f(\bigcap \{g_i(G_p): i=1, \cdots\}) = \bigcap \{f(g_i(G_p)): i=1, \cdots\} = \bigcap \{g_i(G_p): i=1, \cdots\}$. Hence x is divisible; this implies $x=1$ since G is totally indivisible.

The reader should observe that in general a characteristic subloop need not be normal. In the present situation, however, we have actually more specific information:

PROPOSITION 3.5. *If G, G_p, and H_p are as in* 3.4, *then G is the direct product of G_p and H_p.*

Proof. Obviously, $G_p \cap H_p = \{1\}$. We prove next, that $G = G_p H_p$. Let $x \in G$; the group $G(x)$ has a torsion group as character group which decomposes into a direct sum of its p-primary component and a unique complementary summand; correspondingly $G(x)$ is the direct product of its p-indivisible part and its p-divisible part [7, p. 55], the first is $G(x) \cap G_p$ and the latter $G(x) \cap H_p$; hence $G(x) \subset G_p H_p$ and $x \in G_p H_p$. Let now $x = uv$ be a decomposition of an element $x \in G$ with $u \in G_p$, $v \in H_p$, and suppose that we also have $x = u'v'$ with $u' \in G_p$, $v' \in H_p$. We shall show that $u = u'$ and $v = v'$. Consider the group $G(u)G(v)$; if $G(u)$ has, in view of Lemma 1.4, a nontrivial q-group or a q-adic group (q prime) as a direct factor, then $q = p$ since $G(u)$ is in G_p and G_p is divisible by all primes other than p; if, on the other hand, $G(v)$ has a q-group or a q-adic group as a direct factor, then $q \neq p$ since $G(v)$ is in H_p and H_p is divisible by p. Lemma 1.4 now yields the existence of an element y such that $G(y) = G(u)G(v)$. Likewise there is a y' such that $G(y') = G(u')G(v')$. By di-associativity, $G(y)G(y')$ is a group which contains x, u, v, u', and v'.

But in a group the decomposition is unique, so that indeed $u=u'$ and $v=v'$. Next we prove that $x=u(x)v(x)$ and $y=u(y)v(y)$ with $u(x)$, $u(y)\in G_p$ and $v(x)$, $v(y)\in H_p$ implies $(u(x)v(x))(u(y)v(y))=(u(x)u(y))(v(x)v(y))$; but if we pick elements s and t along the lines of the previous procedure so that $G(s)=G(u(x))G(v(x))$ and $G(t)=G(u(y))G(v(y))$, then $G(s)G(t)$ is a group which contains $u(x)$, $v(x)$, $u(y)$, and $v(y)$, hence the asserted identity holds in this abelian group. This observation finishes the proof of the proposition.

We have, in particular, proved that G_p is a characteristic normal subgroup of G. This leads to the following definition:

DEFINITION 3.6. If G is a compact totally indivisible di-associative commutative loop, then the closed normal characteristic subgroup G_p is called the p-indivisible component of G.

Typical groups which could occur as p-indivisible components are finite p-groups, p-adic groups, products of such, in short: all character groups of discrete p-groups. The following is the theorem announced previously which finishes, to some degree, the structure theory for compact commutative di-associative loops:

THEOREM III. *A compact totally indivisible di-associative commutative loop is the direct product of its p-indivisible components.*

Proof. First we observe that the theorem is true for groups because the character group $G^\frown$ of G is a torsion group which splits into a direct sum of its p-primary components. Dually, G is the direct product of the character groups of the primary components of $G^\frown$ which are the p-indivisible components of G. (See [7, p. 55].) According to Proposition 3.5, there is a continuous and open projection $h_p\colon G\to G_p$ with kernel H_p. The mapping $x\to(h_p(x): p=p_1, p_2, \cdots)$, where $p_1, p_2, \cdots$ is the increasing sequence of all primes, is a representation of G onto a compact subloop of the direct product $\prod\{G_p: p=p_1, \cdots\}$. Let now $(x_p: p=p_1, \cdots)$ be an arbitrary element of this product. We define an ascending chain of compact subgroups $G(z_i)$, $i=1, \cdots$ of G by the following recursive process: $G(z_1)=G(x_1)$, $G(z_{n+1})=G(z_n)G(x_{n+1})$; it is easy to see that this construction is possible with the help of Lemma 1.4. Let H be the closure of the union of all $G(z_i)$; then H is a compact subgroup. This subgroup contains all the elements x_p, $p=p_1, \cdots$, and therefore all the elements $y_1=x_{p_1}$, $y_2=x_{p_1}x_{p_2}$, $y_3=x_{p_1}x_{p_2}x_{p_3}$, $\cdots$. In the compact group H this sequence has a subnet $(y_{n(i)}: i\in I)$ which converges to an element $x\in H$. Because of the continuity of h_p we have $\lim_{i\in I} h_p(y_{n(i)})=h_p(x)$ for all p. But for all $i\in I$ such that $p<p_{n(i)}$ we have $h_p(y_{n(i)})=x_p$; consequently $h_p(x)=x_p$. Hence the representation $x\to(h_p(x): p=p_1, \cdots)$ is a representation of G onto $\prod\{G_p: p=p_1, \cdots\}$. The kernel is the set of all $x\in G$ such that $h_p(x)=1$ for all p, i.e. x is in the kernel H_p of all homomorphisms h_p. The intersection of all H_p is the set of divisible elements and is, therefore, $\{1\}$ since G is totally indivisible. Thus the representation is a one

to one representation of the compact loop G onto the compact product loop $\prod\{G_p: p=p_1, \cdots\}$ and is, consequently, an isomorphism. This proves the theorem.

Combining the information contained in Theorem II and Theorem III it becomes apparent that the essentially non-associative features of a compact commutative di-associative loop must be condensed into the indivisible components of the factor loop G/D. The component $(G/D)_3$, e.g., could be Bol's notorious commutative, non-associative Moufang loop with 81 elements and exponent 3 (see [1]). In the case of Moufang loops this comes close to the worst case that can happen, for in any commutative Moufang loop the mapping $x \rightarrow x^3$ is an endomorphism into the center [2]. This implies that all p-indivisible components are groups with the possible exception of $p=3$.

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